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STRESS ANALYSIS IN A JOINT WITH A FUNCTIONALLY GRADED MATERIAL UNDER A THERMAL LOADING BY USING THE MELLIN TRANSFORM METHOD

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Abstract—In a two dissimilar materials joint, near the edges of the interface, very high stresses exist after a homogeneous change of temperature due to the different elastic constants and the different thermal expansion coefficients. In most cases, stress singularities occur for elastic material behaviour. To avoid the stress singularities, a continuous transition in the material properties can be introduced. In a joint with such a functionally graded material (FGM), the finite element method (FEM) is generally used to calculate the stress distribution. In the middle of a thin joint with a graded material the stresses can be calculated analytically by using the beam theory or the plate theory. For a thick joint or in the edge range of a joint no analytical form has been found so far to describe the stresses. In this paper the Mellin transform method is used to describe analytically the stresses in the edge range of a joint with a graded material. Four examples will be presented to show the good agreement between the stresses calculated from FEM and the analytical description in a joint with a graded material under a thermal loading. © 1998 Elsevier Science Ltd.

NOTATION

Φ	stress function
T	temperature change
q	see eqn (1)
E	Young's modulus
G	shear modulus
G_0	shear modulus at $\theta = 0$
ν	Poisson's ratio
α	thermal expansion coefficient
r	polar coordinates
$\bar{r}$	r/L, L is a characteristic length of a joint
θ	polar coordinates
R_0	radius with temperature change in a semi-infinite joint
$\bar{R}_0$	R_0/L
T_0	temperature change in a semi-infinite joint
$\hat{T}$	temperature change in Mellin domain, see eqn (10)
s	Mellin transform parameter
σ_{ij}	stress tensor
σ	normal stress component
τ	shear stress component
u	displacement component in the r direction
v	displacement component in the θ direction
m and κ	constants, see eqn (5)
$\bar{u}(s, \theta)$	see eqn (18)
$\bar{v}(s, \theta)$	see eqn (20)
$\hat{T}_n$	temperature change in Mellin domain, see eqn (25)
$\hat{B}$	see eqn (28)
$\tilde{\sigma}_{ijk}(r, \theta)$	see eqns (69), (70) and (71)
s_n	pole of $\hat{\sigma}_{ij}(s, \theta)$
ω_n	stress exponent, $\omega_n = -(s_n+2)$

INTRODUCTION

If two dissimilar materials are joined at high temperature, very high residual stresses develop near the free edge of the interface during cooling to room temperature. In most cases, the stresses are singular for elastic material behaviour. These stresses may cause failure directly after cooling or after a small amount of mechanical loading. In order to reduce the residual stresses, an interlayer can be introduced between the two materials. By adapting the properties of the interlayer to those of the joined components and by varying the thickness of the interlayer, the residual stresses can be greatly reduced compared to those occurring when two materials were joined directly [see Munz *et al.* (1995)]. However, due to the discontinuity of the material properties at the interfaces, in most cases, there are still stress singularities near the free edge of the interface. The stress singularity only gets weaker by the introduction of a favourable interlayer.

Another way of reducing the residual stresses is the application of a graded interfacial zone with a continuous change of all material properties. In the last 10 years many investigations have been performed with regard to the graded materials, which are called functionally graded material (FGM). These investigations have concentrated on the following aspects: (a) manufacturing of FGM [see Nagano and Wakai (1993)]; (b) design of FGM [see Hirano *et al.* (1990)]; (c) describing the stress distribution in FGM [see Arai *et al.* (1990); Williamson *et al.* (1993)]; (d) stress field at the tip of the crack in FGM [see Erdogan (1990)] etc. Up to now, for the description of the stresses in a joint with a graded material only the finite element method (FEM) has been used [see Arai *et al.* (1990); Williamson *et al.* (1993); Yang and Munz (1995)].

In this paper, the Mellin transform method is used to describe analytically the stresses near the free edge of the interface in a joint with a graded material under a thermal loading. Four examples will be presented to show the good agreement between the stresses calculated from FEM and the analytical description in a joint with a graded material.

THE BASIC EQUATIONS

When a thermal loading is taken into account and body forces are disregarded, the stress function Φ in a homogeneous material should satisfy the equation

$$\nabla^4\Phi+\nabla^2(qT) = 0 \tag{1}$$

with

$$q = \begin{cases} \alpha E & \text{for plane stress} \\ \dfrac{\alpha E}{(1-\nu)} & \text{for plane strain} \end{cases}$$

where T is the temperature change, E Young's modulus, ν Poisson's ratio, α thermal expansion coefficient and q is a constant. In a graded material, in which the material properties are dependent on the coordinates, q is not a constant, but $q = q(x, y) = q(\bar{r}, \theta)$. We can imagine that qT is the effective temperature change and the material is homogeneous. Under this assumption, for a graded material we obtain the same equation like eqn (1) for the stress function. It should be noted that for the boundary conditions the real inhomogeneous material properties should be considered.

The relations between the stress component and the stress function $\Phi(\bar{r}, \theta)$ are

$$\sigma_{rr} = \frac{\partial\Phi}{\bar{r}\,\partial\bar{r}}+\frac{\partial^2\Phi}{\bar{r}^2\,\partial\theta^2} \tag{2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial\bar{r}^2} \tag{3}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial\bar{r}}\left(\frac{\partial\Phi}{\bar{r}\,\partial\theta}\right) \tag{4}$$

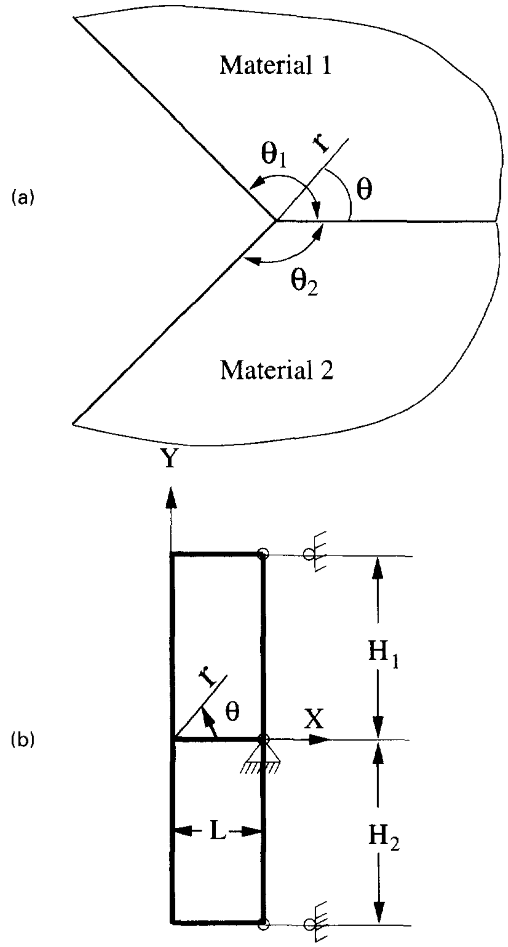

Fig. 1. The geometry investigated.

where r and θ are the polar coordinates (see Fig. 1). The introduction of a dimensionless quantity $\bar{r} = r/L$ is for the convenience of the Mellin transform, where L is a characteristic length of the joint (see Fig. 1b).

By using the Hook's law between the stresses and the strains, and the relations between the strains and the displacements, the displacements $u = u(\bar{r}, \theta)$ and $v = v(\bar{r}, \theta)$ in r and θ direction can be calculated from the stress function as follows:

$$\frac{\partial u}{\partial \bar{r}} = \frac{1}{2G}\left\{\frac{\partial \Phi}{\bar{r}\,\partial \bar{r}} + \frac{\partial^2 \Phi}{\bar{r}^2\,\partial \theta^2} - \left(1 - \frac{m}{4}\right)\nabla^2 \Phi\right\} + \frac{q}{E}\kappa T \tag{5}$$

$$\frac{\partial u}{\bar{r}\,\partial \theta} + \frac{\partial v}{\partial \bar{r}} - \frac{v}{\bar{r}} = \frac{1}{G}\left\{\frac{\partial \Phi}{\bar{r}^2\,\partial \theta} - \frac{\partial^2 \Phi}{\bar{r}\,\partial \bar{r}\,\partial \theta}\right\} \tag{6}$$

with

$$m = \begin{cases} \dfrac{4}{(1+\nu)} & \text{for plane stress} \\ 4(1-\nu) & \text{for plane strain} \end{cases}$$

and

$$\kappa = \begin{cases} 1 & \text{for plane stress} \\ (1-\nu^2) & \text{for plane strain} \end{cases}.$$

If a stress function, which satisfies eqn (1) and the boundary conditions, is found, the stresses in a joint with a graded material can be described analytically. The aim of this paper is to find a solution of such a stress function and to determine all its unknown coefficients. The Mellin transform method is used to solve this kind of problem, because in Mellin domain eqn (1) is replaced by an ordinary differential equation and it is much easier to find a general solution of an ordinary differential equation. In the Mellin transform method a semi-infinite space is considered. Therefore, at first a semi-infinite joint with a graded material under the temperature change

$$T = \begin{cases} T_0 & \text{for } r \leqslant R_0 \\ 0 & \text{for } r > R_0 \end{cases}$$

is treated.

SOLUTION IN THE MELLIN TRANSFORM DOMAIN

The Mellin transform of a function $\Phi(\bar{r}, \theta)$ is defined as [see Dautray and Lions (1988)]

$$\hat{\Phi}(s, \theta) = \int_0^{\infty} \Phi(\bar{r}, \theta)\bar{r}^{(s-1)}\,\mathrm{d}\bar{r} \tag{7}$$

where s is the parameter of the Mellin transform. The parameter s should be chosen such that the integration in eqn (7) is valid. The property of the Mellin transform is

$$\int_0^{\infty} \bar{r}^p \frac{\partial^q \Phi(\bar{r}, \theta)}{\partial \bar{r}^q} \bar{r}^{s-1}\,\mathrm{d}\bar{r} = (-1)^q \frac{\Gamma(s+p)}{\Gamma(s+p-q)} \hat{\Phi}(s+p-q, \theta) \tag{8}$$

where $\Gamma(x)$ is the Γ-function.

The Mellin transform of eqn (1) then reads:

$$\left[s^2 + \frac{\partial^2}{\partial \theta^2}\right]\left[(s+2)^2 + \frac{\partial^2}{\partial \theta^2}\right]\hat{\Phi}(s, \theta) + \left[(s+2)^2 + \frac{\partial^2}{\partial \theta^2}\right]\hat{T}(s+2, \theta) = 0 \tag{9}$$

with

$$\hat{T}(s+2, \theta) = \int_0^{\infty} q(\bar{r}, \theta) T \bar{r}^{(s+1)}\,\mathrm{d}\bar{r} = T_0 \int_0^{\bar{R}_0} q(\bar{r}, \theta)\bar{r}^{(s+1)}\,\mathrm{d}\bar{r} \tag{10}$$

and $\bar{R}_0 = R_0/L$. Here, L has no specific meaning. For a finite joint, it is a characteristic length of the joint.

If s is considered as a parameter, eqn (9) is an ordinary differential equation of the variable θ. For each value of s, its solution is

$$\hat{\Phi}_k(s, \theta) = \hat{\Phi}_k^{\mathrm{I}}(s, \theta) + \hat{\Phi}_k^{\mathrm{II}}(s, \theta) \tag{11}$$

with

$$\hat{\Phi}_k^{\mathrm{I}}(s, \theta) = A_k\,\mathrm{e}^{is\theta} + \bar{A}_k\,\mathrm{e}^{-is\theta} + B_k\,\mathrm{e}^{i(s+2)\theta} + \bar{B}_k\,\mathrm{e}^{-i(s+2)\theta} \tag{12}$$

and

$$\hat{\Phi}_k^{\mathrm{II}}(s,\theta) = -\frac{1}{s}\left\{\sin(s\theta)\int \hat{T}_k(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta - \cos(s\theta)\int \hat{T}_k(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\right\} \tag{13}$$

where A_k, B_k ($\bar{A}_k$, $\bar{B}_k$ is the conjugate complex number of A_k, B_k) are unknown complex coefficients and $k = 1, 2$ for materials 1 and 2. The function $\hat{\Phi}_k^{\mathrm{I}}(s,\theta)$ is the general solution of the homogeneous part of eqn (9), which is dependent only on the material properties and the joint geometry, independent on the loading. The function $\hat{\Phi}_k^{\mathrm{II}}(s,\theta)$ is the special solution of eqn (9), which is dependent on the loading. If the coefficients A_k and B_k are determined, the solution of the problem in the Mellin domain is known. In order to determine the unknown coefficients A_k and B_k, the boundary conditions expressed by the Mellin transform have to be used. For a joint with free edge they are:

for the free edges

$$\begin{aligned} \theta = \theta_1 &: \hat{\tau}_{r\theta 1} + i\hat{\sigma}_{\theta\theta 1} = 0 \\ \theta = \theta_2 &: \hat{\tau}_{r\theta 2} + i\hat{\sigma}_{\theta\theta 2} = 0 \end{aligned} \tag{14}$$

at the interface

$$\begin{aligned} \theta = 0 &: \hat{\tau}_{r\theta 1} + i\hat{\sigma}_{\theta\theta 1} = \hat{\tau}_{r\theta 2} + i\hat{\sigma}_{\theta\theta 2} \\ \theta = 0 &: \hat{u}_1 + i\hat{v}_1 = \hat{u}_2 + i\hat{v}_2. \end{aligned} \tag{15}$$

The physical meaning of eqn (14) is that, at the free edges $\theta = \theta_1$ and $\theta = \theta_2$, the normal stress σ_θ and the shear stress $\tau_{r\theta}$ are zero. The physical meaning of eqn (15) is that at the interface $\theta = 0$ the normal stress σ_θ, the shear stress $\tau_{r\theta}$ and the displacements u and v should be continuous in materials 1 and 2. To use the boundary conditions the stresses and the displacements have to be transformed in the Mellin domain.

From the eqns (2)–(4) we obtain the Mellin transform of the stress components as follows:

$$\begin{aligned} \hat{\sigma}_{rr}(s,\theta) &= \left(\frac{\partial^2}{\partial\theta^2} - s\right)\hat{\Phi}(s,\theta) \\ \hat{\sigma}_{\theta\theta}(s,\theta) &= (s+1)s\hat{\Phi}(s,\theta) \\ \hat{\tau}_{r\theta}(s,\theta) &= (s+1)\frac{\partial\hat{\Phi}(s,\theta)}{\partial\theta}. \end{aligned} \tag{16}$$

The Mellin transform of the displacement u can be obtained from eqn (5) as:

$$\begin{aligned} \hat{u}(s+1,\theta) &= -\frac{1}{(s+1)}\int_0^\infty \bar{r}^{(s+1)}\left[\frac{1}{2G}\left\{\frac{\partial\Phi}{\bar{r}\,\partial\bar{r}} + \frac{\partial^2\Phi}{\bar{r}^2\,\partial\theta^2} - \left(1-\frac{m}{4}\right)\nabla^2\Phi\right\} + \frac{q}{E}\kappa T\right]\mathrm{d}\bar{r} \\ &= \bar{u}(s,\theta) \end{aligned} \tag{17}$$

with

$$\bar{u}(s,\theta) = -\frac{1}{(s+1)}\int_0^\infty \bar{r}^{(s+1)}\left[\frac{1}{2G}\left\{\frac{\partial\Phi}{\bar{r}\,\partial\bar{r}} + \frac{\partial^2\Phi}{\bar{r}^2\,\partial\theta^2} - \left(1-\frac{m}{4}\right)\nabla^2\Phi\right\} + \frac{q}{E}\kappa T\right]\mathrm{d}\bar{r}. \tag{18}$$

The Mellin transform of the displacement v can be calculated from:

$$\frac{\partial \hat{u}(s+1,\theta)}{\partial \theta}-(s+2)\hat{v}(s+1,\theta) = \int_0^\infty \bar{r}^{(s+1)}\left[\frac{1}{G}\left\{\frac{\partial \Phi}{\bar{r}^2\,\partial \theta}-\frac{\partial^2 \Phi}{\bar{r}\,\partial \bar{r}\,\partial \theta}\right\}\right]\mathrm{d}\bar{r}$$
$$= \bar{v}(s,\theta) \tag{19}$$

with

$$\bar{v}(s,\theta) = \int_0^\infty \bar{r}^{(s-1)}\left[\frac{1}{G}\left\{\frac{\partial \Phi}{\bar{r}^2\,\partial \theta}-\frac{\partial^2 \Phi}{\bar{r}\,\partial \bar{r}\,\partial \theta}\right\}\right]\mathrm{d}\bar{r} \tag{20}$$

which is the Mellin transform of eqn (6).

The complex form of the stresses and displacements in the Mellin domain can be given as

$$\hat{\tau}_{r\theta}(s,\theta)+i\hat{\sigma}_{\theta\theta}(s,\theta) = (s+1)\left(\frac{\partial}{\partial \theta}+is\right)\hat{\Phi}(s,\theta) \tag{21}$$

and

$$\hat{u}(s+1,\theta)+i\hat{v}(s+1,\theta) = \bar{u}(s,\theta)+i\frac{1}{(s+2)}\left[\frac{\partial \bar{u}(s,\theta)}{\partial \theta}-\bar{v}(s,\theta)\right]. \tag{22}$$

For using the boundary conditions [see eqns (14) and (15)] we need the values of $\bar{u}(s,\theta)$ in eqn (18) and $\bar{v}(s,\theta)$ in eqn (20) at the interface $\theta = 0$. We assume that at $\theta = 0$ the shear modulus G is a constant, but $\partial G/\partial \theta \neq 0$. This is, for instance, the case at the interface of a joint with one homogeneous material and one graded material. Because the variation of the Poisson's ratio ν in a graded material has a small effect on the stresses [see Eischen (1987)], we assume that in a graded material the Poisson's ratio is a constant. Then from eqn (18) we have

$$\bar{u}(s,\theta)\Big|_{\theta=0} = \left[\frac{1}{2G_0}\left\{s-\frac{ms^2}{4(s+1)}-\frac{m}{4(s+1)}\frac{\partial^2}{\partial \theta^2}\right\}\hat{\Phi}(s,\theta)-\frac{\hat{T}_n(s+2,\theta)}{(s+1)}\right]\Bigg|_{\theta=0} \tag{23}$$

$$\frac{\partial \bar{u}(s,\theta)}{\partial \theta}\Big|_{\theta=0} = -\frac{1}{(s+1)}\left\{-\frac{1}{2G_0^2}\left[\int_0^\infty \bar{r}^{(s+1)}\frac{\partial G}{\partial \theta}\Big|_{\theta=0}\left\{\frac{\partial \Phi}{\bar{r}\,\partial \bar{r}}+\frac{\partial^2 \Phi}{\bar{r}^2\,\partial \theta^2}-\left(1-\frac{m}{4}\right)\nabla^2\Phi\right\}\mathrm{d}\bar{r}\right]\Bigg|_{\theta=0}\right.$$
$$\left.+\frac{1}{2G_0}\left[\frac{\partial}{\partial \theta}\int_0^\infty\left\{\frac{\partial \Phi}{\bar{r}\,\partial \bar{r}}+\frac{\partial^2 \Phi}{\bar{r}^2\,\partial \theta^2}-\left(1-\frac{m}{4}\right)\nabla^2\Phi\right\}\mathrm{d}\bar{r}\right]\Bigg|_{\theta=0}+\frac{\partial \hat{T}_n(s+2,\theta)}{\partial \theta}\Big|_{\theta=0}\right\} \tag{24}$$

with

$$\hat{T}_n(s+2,\theta) = \int_0^\infty \frac{q(\bar{r},\theta)}{E}\kappa T\bar{r}^{(s+1)}\,\mathrm{d}\bar{r} = T_0\int_0^{\bar{R}_0}\frac{q(\bar{r},\theta)}{E}\kappa\bar{r}^{(s+1)}\,\mathrm{d}\bar{r} \tag{25}$$

and

$$G_0 = G\,|_{\theta=0}. \tag{26}$$

From eqn (20) we have

$$\bar{v}(s,\theta)|_{\theta=0} = \frac{1}{G_0}\left[(s+1)\frac{\partial \hat{\Phi}(s,\theta)}{\partial \theta}\right]\Bigg|_{\theta=0}. \tag{27}$$

From eqns (22) and (24) it can be seen that the solution of the problem is strongly dependent on the profile of the shear modulus G.

In the general case, i.e. the function form of the shear modulus G is arbitrary, to determine the unknown coefficients A_k and B_k in eqn (12) we need to substitute eqn (11) into eqns (21)–(24) and (27), and into eqns (14) and (15). The stresses and displacements in the Mellin domain can then be calculated from eqns (16), (17) and (19). The solution of the problem is dependent on the function form of the material properties (G, E, v, α). The more complicated the function form of G, E and α is, the more difficult it is to obtain the analytical description of the stresses near the free edge of the interface in a joint with a graded material. It is impossible to give a generally explicit expression of the coefficients A_k and B_k as a function of the material properties, the joint geometry and the loading for an arbitrary transition profile in the graded material. Here, only a general and possible procedure is given to obtain an analytical description of the stresses near the free edge of the interface in a joint with a graded material.

In the following we consider the case, in which the graded material has the profile of a polynomial and it is only dependent on the coordinate y, i.e.

$$\begin{aligned} G = \frac{E(\bar{r},\theta)}{2(1+v)} &= [\check{A}+\check{B}\bar{y}+\check{C}\bar{y}^2+\check{D}\bar{y}^3+\check{E}\bar{y}^4+\cdots] \\ &= [\check{A}+\check{B}\bar{r}\sin(\theta)+\check{C}\bar{r}^2\sin^2(\theta)+\check{D}\bar{r}^3\sin^3(\theta)+\check{E}\bar{r}^4\sin^4(\theta)+\cdots] \end{aligned} \tag{28}$$

and

$$\left.\frac{\partial G}{\partial \theta}\right|_{\theta=0} = \check{B}\bar{r} \tag{29}$$

where there is $\bar{y} = y/L$. For this case an explicit expression for the coefficients A_k and B_k can be given as a function of the material properties, the joint geometry and the loading.

In the boundary conditions only the displacements are related to the material property. For the profile given in eqn (28) the displacement condition can be obtained by insetting eqn (29) into eqn (23), eqn (24), eqn (27) and then into eqn (22), it is:

$$\begin{aligned} &2G_0[\hat{u}(s+1,\theta)+i\hat{v}(s+1,\theta)]\Big|_{\theta=0} \\ &= \left(s-i\frac{\partial}{\partial\theta}\right)\left\{\left[1+\frac{m\left(\frac{\partial}{\partial\theta}-is\right)\left(\frac{\partial}{\partial\theta}-i(s+2)\right)}{4(s+1)(s+2)}\right]\hat{\Phi}(s,\theta)\right\}\Bigg|_{\theta=0} - \frac{2G_0\hat{T}_n(s+2,\theta)}{(s+1)}\Bigg|_{\theta=0} \\ &\quad + i\left\{\frac{\check{B}}{G_0}\left[\frac{m}{4(s+1)(s+2)}\frac{\partial^2\hat{\Phi}(s+1,\theta)}{\partial\theta^2} - \left(1-\frac{m(s+1)}{4(s+2)}\right)\hat{\Phi}(s+1,\theta)\right]\Bigg|_{\theta=0}\right. \\ &\quad \left. - \frac{2G_0}{(s+1)(s+2)}\frac{\partial\hat{T}_n(s+2,\theta)}{\partial\theta}\Bigg|_{\theta=0}\right\}. \end{aligned} \tag{30}$$

To give an explicit expression of the unknown coefficients A_k, $\bar{A}_k$, B_k and $\bar{B}_k$ in eqn (12), they are separated into real and imaginary parts as

$$\begin{aligned} A_k &= C_k+iD_k \quad & \bar{A}_k &= C_k-iD_k \\ B_k &= F_k+iH_k \quad & \bar{B}_k &= F_k-iH_k. \end{aligned} \tag{31}$$

By substituting eqn (11) in eqns (21) and (30) and then in the boundary conditions eqns (14) and (15), and separating the real and imaginary parts of each equation, the coefficients C_k, D_k, F_k and H_k can be determined.

In the following one special case with $\theta_1 = -\theta_2$ will be treated and the explicit expressions for the calculation of the coefficients C_k, D_k, F_k and H_k will be given. After simplifying and solving the equations from the real and imaginary part, relations for the coefficients C_k, D_k, F_k and H_k can be obtained. They are:

$$C_1 = \frac{C_1^*}{(s+s_n)\Delta_2\Delta^*} \tag{32}$$

$$C_2 = \frac{C_2^*}{(s+s_n)\Delta_2\Delta^*} \tag{33}$$

$$D_1 = \frac{D_1^*}{(s+s_n)\Delta_2\Delta^*} \tag{34}$$

$$D_2 = \frac{D_2^*}{(s+s_n)\Delta_2\Delta^*} \tag{35}$$

$$F_1 = F_2 = F = \frac{F^*}{(s+s_n)\Delta_2\Delta^*} \tag{36}$$

$$H_1 = \frac{H_1^*}{(s+s_n)\Delta_2\Delta^*} \tag{37}$$

$$H_2 = \frac{H_2^*}{(s+s_n)\Delta_2\Delta^*} \tag{38}$$

with

$$F^* = t_1^* a_{22}\Delta_5 - t_2^* a_{12}\Delta_2 \tag{39}$$

$$H_1^* = t_2^* a_{11}\Delta_5 - t_1^* a_{21}^* \tag{40}$$

$$H_2^* = R_1^*\Delta^* - H_1^* \tag{41}$$

$$C_1^* = \frac{1}{s}\left\{F^*\Delta_4 + H_1^*\Delta_2 - \frac{(s+s_n)\Delta^*\Delta_2}{2}\int \hat{T}_1(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\bigg|_{\theta=\theta_1}\right\} \tag{42}$$

$$C_2^* = \frac{1}{s}\left\{F^*\Delta_4 + H_1^*\Delta_2 - R_1^*\Delta_2\Delta^* - \frac{(s+s_n)\Delta^*\Delta_2}{2}\int \hat{T}_2(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\bigg|_{\theta=\theta_2}\right\} \tag{43}$$

$$D_1^* = \frac{1}{s}\left\{-F^*\Delta_1 - H_1^*\Delta_3 - \frac{(s+s_n)\Delta^*\Delta_2}{2}\int \hat{T}_1(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta\bigg|_{\theta=\theta_1}\right\} \tag{44}$$

$$D_2^* = \frac{1}{s}\left\{F^*\Delta_1 + H_1^*\Delta_3 - R_1^*\Delta_3\Delta^* - \frac{(s+s_n)\Delta^*\Delta_2}{2}\int \hat{T}_2(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta\bigg|_{\theta=\theta_2}\right\} \tag{45}$$

and

$$a_{11} = 2\Delta_1 \tag{46}$$

$$a_{12} = 2(\Delta_3 - (s+2)) \tag{47}$$

$$a_{21} = 2\left\{\left(\Delta_4 + \left(1 + \frac{m}{s+1}\right)s\right)(\breve{B}_2 - \breve{B}_1)\frac{1}{s} + \frac{2Gm\Delta_1}{\Delta_3 - (s+2)}\bigg|_{\theta=0}\right\} \tag{48}$$

$$a_{22} = 2\Delta_2(\breve{B}_2-\breve{B}_1)\frac{1}{s} \tag{49}$$

$$a_{21}^* = 2\left\{\left(\Delta_4+\left(1+\frac{m}{s+1}\right)s\right)(\breve{B}_2-\breve{B}_1)\frac{1}{s}\Delta_5+2G_0 m\Delta_1\right\} \tag{50}$$

$$R_1^* = R_1(s+s_n) \tag{51}$$

$$R_2^* = R_2(s+s_n) \tag{52}$$

$$R_3^* = R_3(s+s_n) \tag{53}$$

$$R_4^* = R_4(s+s_n) \tag{54}$$

$$t_1^* = R_1^*\Delta_5 - R_2^*\Delta_2 \tag{55}$$

$$t_2^* = \left[R_3^*+R_4^*+2R_1^*\breve{B}_2\frac{1}{s}\right]\Delta_5 - 2G_0 m R_2^* \tag{56}$$

with

$$\Delta_1 = \sin(2(s+1)\theta_1)+(s+1)\sin(2\theta_1) \tag{57}$$

$$\Delta_2 = -\sin(2(s+1)\theta_1)+(s+1)\sin(2\theta_1) \tag{58}$$

$$\Delta_3 = \cos(2(s+1)\theta_1)+(s+1)\cos(2\theta_1) \tag{59}$$

$$\Delta_4 = \cos(2(s+1)\theta_1)-(s+1)\cos(2\theta_1) \tag{60}$$

$$\Delta_5 = \Delta_3-(s+2) \tag{61}$$

$$\Delta = a_{11}a_{22}-a_{12}a_{21} \tag{62}$$

$$\Delta^* = \Delta\Delta_5 \tag{63}$$

$$R_1 = \frac{1}{2}\left\{\int_0^{\theta_1}\hat{T}_1(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta - \int_0^{\theta_2}\hat{T}_2(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\right\} \tag{64}$$

$$R_2 = \frac{1}{2}\left\{\int_0^{\theta_1}\hat{T}_1(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta - \int_0^{\theta_2}\hat{T}_2(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta\right\} \tag{65}$$

$$R_3 = \frac{1}{s}\left\{\breve{B}_2\int\hat{T}_2(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\bigg|_{\theta=\theta_2} - \breve{B}_1\int\hat{T}_1(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\bigg|_{\theta=\theta_1}\right\} \tag{66}$$

$$\begin{aligned}R_4 = {} & \frac{mG_0}{4(s+1)(s+2)}\left[\frac{\partial\hat{T}_2(s+2,\theta)}{\partial\theta}-\frac{\partial\hat{T}_1(s+2,\theta)}{\partial\theta}\right]\bigg|_{\theta=0} \\ & +\frac{2G_0^2}{(s+1)(s+2)}\left[\frac{\partial\hat{T}_{1n}(s+2,\theta)}{\partial\theta}-\frac{\partial\hat{T}_{2n}(s+2,\theta)}{\partial\theta}\right]\bigg|_{\theta=0} \\ & +\breve{B}_1\left\{\frac{1}{(s+1)}\int\hat{T}_1(s+3,\theta)\sin((s+1)\theta)\,\mathrm{d}\theta\bigg|_{\theta=0}+\frac{m}{4(s+1)(s+2)}\hat{T}_1(s+3,\theta)\bigg|_{\theta=0}\right\}\end{aligned}$$

$$-\breve{B}_2\left\{\frac{1}{(s+1)}\int\hat{T}_2(s+3,\theta)\sin((s+1)\theta)\,\mathrm{d}\theta\bigg|_{\theta=0}+\frac{m}{4(s+1)(s+2)}\hat{T}_2(s+3,\theta)\bigg|_{\theta=0}\right\} \tag{67}$$

where s_n is the solution of the equation $\Delta_2\Delta^* = 0$.

Finally, the stresses in the Mellin domain can be calculated from:

$$\hat{\sigma}_{ijk}(s,\theta)=\frac{\tilde{\sigma}_{ijk}(s,\theta)}{(s+s_n)\Delta^*\Delta_2} \tag{68}$$

with

$$\begin{aligned}\tilde{\sigma}_{rrk}(s,\theta) = &-C_k^*2s(s+1)\cos(s\theta)+D_k^*2s(s+1)\sin(s\theta)\\ &-F_k^*2(s^2+5s+4)\cos((s+2)\theta)+H_k^*2(s^2+5s+4)\sin((s+2)\theta)\\ &+(s+s_n)\Delta^*\Delta_2(s+1)\left[\sin(s\theta)\int\hat{T}_k(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta\right.\\ &\left.-\cos(s\theta)\int\hat{T}_k(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\right]-(s+s_n)\Delta^*\Delta_2\hat{T}_k(s+2,\theta)\end{aligned} \tag{69}$$

$$\begin{aligned}\tilde{\sigma}_{\theta\theta k}(s,\theta) = s(s+1)&\left\{C_k^*2\cos(s\theta)-D_k^*2\sin(s\theta)+F_k^*2\cos((s+2)\theta)+H_k^*2\sin((s+2)\theta)\right.\\ &\left.-\frac{1}{s}(s+s_n)\Delta^*\Delta_2\left[\sin(s\theta)\int\hat{T}_k(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta-\cos(s\theta)\int\hat{T}_k(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\right]\right\}\end{aligned} \tag{70}$$

$$\begin{aligned}\tilde{\sigma}_{r\theta k}(s,\theta) = -2(s+1)&\left\{C_k^*s\sin(s\theta)+D_k^*s\cos(s\theta)\right.\\ &+F_k^*(s+2)\sin((s+2)\theta)+H_k^*(s+2)\cos((s+2)\theta)\\ &\left.+\frac{1}{2}(s+s_n)\Delta^*\Delta_2\left[\cos(s\theta)\int\hat{T}_k(s+2,\theta)\cos(s\theta)\,\mathrm{d}\theta+\sin(s\theta)\int\hat{T}_k(s+2,\theta)\sin(s\theta)\,\mathrm{d}\theta\right]\right\}\end{aligned} \tag{71}$$

where the coefficients C_k^*, D_k^*, F_k^*, H_k^* are known from eqns (39)–(45), and $k = 1$ and 2 is for materials 1 and 2.

SOLUTION IN THE POLAR COORDINATE SYSTEM

Our aim is to calculate the stresses in a polar coordinates system, i.e. $\sigma_{ijk}(s,\theta)$, which is the reversal transform of $\hat{\sigma}_{ijk}(s,\theta)$. For the calculation of the reversal of $\hat{\sigma}_{ijk}(s,\theta)$ we need the poles of $\hat{\sigma}_{ijk}(s,\theta)$, which are defined as follows: if $\lim_{s\to s_n}\hat{\sigma}_{ijk}(s,\theta)\to\infty$, s_n is the pole of $\hat{\sigma}_{ijk}(s,\theta)$. From eqn (68) it can be seen that the possible poles of $\hat{\sigma}_{ijk}(s,\theta)$ are the solutions of $\Delta_2\Delta^* = 0$.

From the definition of the reversion of the Mellin transform, the stresses in a polar coordinate system can be calculated by

$$\sigma_{ij}(\bar{r},\theta)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\hat{\sigma}_{ij}(s,\theta)\bar{r}^{-(s+2)}\,\mathrm{d}s \tag{72}$$

where γ must be chosen so that the integration in eqn (72) exists. According to the residual principle the stresses in the polar coordinate system can be calculated by

$$\sigma_{ij}(r,\theta) = \sum_{s_n<\gamma} \mathrm{res}\{\hat{\sigma}_{ij}(s_n,\theta)\bar{r}^{-(s_n+2)}\}$$

$$= \sum_{s_n<\gamma} \frac{1}{(M-1)!}\lim_{s\to s_n}\frac{\mathrm{d}^{M-1}}{\mathrm{d}s^{M-1}}\{(s-s_n)^M\hat{\sigma}_{ij}(s,\theta)\bar{r}^{-(s+2)}\}$$

$$= \sum_{s_n<\gamma} \frac{1}{(M-1)!}\lim_{s\to s_n}\frac{\mathrm{d}^{M-1}}{\mathrm{d}s^{M-1}}\left\{(s-s_n)^M\frac{\tilde{\sigma}_{ij}(s,\theta)}{(s+s_n)\Delta^*\Delta_2}\bar{r}^{-(s+2)}\right\} \quad (73)$$

where s_n is the pole of Mth order of $\hat{\sigma}_{ij}(s,\theta)$.

If s_n is the pole of the first-order of $\hat{\sigma}_{ij}(s,\theta)$, the corresponding stress term has the form as follows:

$$\sigma_{ijn}(r,\theta) = \bar{r}^{\omega_n}[f_{ijn}(\theta,K_n)+h_{ijn}(\theta)] \quad (74)$$

with $\omega_n = -(s_n+2)$. If s_n is the pole of the second-order of $\hat{\sigma}_{ij}(s,\theta)$, the corresponding stress term is

$$\sigma_{ijn}(r,\theta) = \bar{r}^{\omega_n}[f_{ijn}(\theta,K_n)+g_{ijn}(\theta)\ln(\bar{r})+h_{ijn}(\theta)]. \quad (75)$$

In eqns (74) and (75) the term $f_{ijn}(\theta,K_n)$ will be calculated from

$$f_{ijn}(\theta,K_n) = K_{n1}F_{ijn1}(\theta)+K_{n2}F_{ijn2}(\theta)+K_{n3}F_{ijn3}(\theta)+,\cdots, \quad (76)$$

where K_{nl} $(l = 1,2,3,\ldots)$ is a function of $\bar{R}_0$ and is dimensionless. How many terms there are in eqn (76) is dependent on the value of ω_n and the order of the pole. ω_n is the stress exponent and $F_{ijnl}(\theta)$, $g_{ijn}(\theta)$ and $h_{ijn}(\theta)$ are the angle functions, which can be determined analytically for each pole s_n by setting eqns (68)–(71) into eqn (73). The angular functions have a unit of the stress, i.e. MPa or GPa.

From eqn (74) we know that if $s_n = -2$ is the pole of the first-order of $\hat{\sigma}_{ij}(s,\theta)$, the corresponding stress term is independent of the coordinate r. Furthermore, as the displacements u and v are proportional to $r^{-(s_n+1)}$, only the solutions with $s_n \leqslant -1$ have a physical meaning (-1 is the γ in eqn (73)). The case with $s = -1$ is corresponding to the rigid body displacement and if $s > -1$, the displacements at $r = 0$ are infinite.

Generally, for each pole s_n of the first- or second-order of $\hat{\sigma}_{ij}(s,\theta)$, the corresponding stress term can be written as follows

$$\sigma_{ijn}(r,\theta) = \bar{r}^{\omega_n}[f_{ijn}(\theta,K_n)+g_{ijn}(\theta)\ln(\bar{r})+h_{ijn}(\theta)]. \quad (77)$$

For the pole of the first-order of $\hat{\sigma}_{ij}(s,\theta)$ there is always $g_{ijn}(\theta) \equiv 0$.

Finally, considering N possible poles of $\hat{\sigma}_{ij}(s,\theta)$, the stresses in a joint with a graded material can be calculated by

$$\sigma_{ij}(r,\theta) = \sum_{n=1}^{N}\{(r/L)^{\omega_n}[f_{ijn}(\theta,K_n)+g_{ijn}(\theta)\ln(r/L)+h_{ijn}(\theta)]\}. \quad (78)$$

In eqns (78) and (76) the angular functions F_{ijnl}, $g_{ijn}(\theta)$, $h_{ijn}(\theta)$ and the stress exponents ω_n can be determined analytically, they are a function of the profile of the shear modulus G and the thermal expansion coefficient α, and also a function of the value of the Poisson's ratio ν and the geometry (angles θ_1,θ_2). For a semi-infinite joint R_0 is given, therefore, K_{nl} is well known. The stresses $\sigma_{ij}(r,\theta)$ is proportional to the temperature change T_0.

In the following, the joint with $\theta_1 = -\theta_2 = 90^\circ$ will be considered more in details. To calculate the quantities F_{ijnl}, $g_{ijn}(\theta)$, $h_{ijn}(\theta)$ and the stress exponents ω_n from eqns (68)–(71)

and (73) at first we need to know the poles of $\hat{\sigma}_{ij}(s_n, \theta)$, which are the solutions of $\Delta_2\Delta_5\Delta = 0$. For $\theta_1 = -\theta_2 = 90°$ from eqn (58) there is

$$\Delta_2 = -\sin((s+1)\pi). \tag{79}$$

It can be seen that $s_n = 0, \pm 1, \pm 2, \pm 3, \ldots, \pm n, \ldots$ are always the solution of the equation $\Delta_2 = 0$. In the range of $s_n \leqslant -1$, $s_n = -2, -3, -4, \ldots, -n, \ldots$, may be the poles of $\hat{\sigma}_{ij}(s, \theta)$.

We also need the solution of $\Delta_5 = 0$. For $\theta_1 = 90°$ we have (see eqn (61))

$$\Delta_5 = \cos[(s+1)\pi] - (s+1) - (s+2) = \cos[(s+1)\pi] - 2s - 3. \tag{80}$$

Because there is always $-1 \leqslant \cos[(s+1)\pi] \leqslant 1$, the solution of $\Delta_5 = 0$ is in the range of $-2 \leqslant s \leqslant -1$. It is known that in a quarter-planes joint with a graded material and a homogeneous material at the interface the material properties are continuous, therefore, there is no stress singularity. This means that the stress exponent ω_n should be greater than zero, i.e. $s_n \leqslant -2$. In the range of $s_n \leqslant -2$, the solution of $\Delta_5 = 0$ is not the pole of $\hat{\sigma}_{ij}(s, \theta)$, except for $s = -2$.

Now, we will look for the solution of the equation $\Delta = 0$. For a joint with one homogeneous material (material 1) and one graded material (material 2) we have $\breve{B}_1 = 0$, and the case I : $\breve{B}_2 = 0$ and the case II : $\breve{B}_2 \neq 0$.

For the case I : $\breve{B}_2 = 0$, Δ is very simple, which is (see eqn (62))

$$\Delta = -8(s+2)mG_0 \sin[(s+1)\pi]. \tag{81}$$

The solution of $\Delta = 0$ is independent of material properties and its solution is also $s_n = -2, -3, -4, \ldots, -n, \ldots$. This means that as long as the polynomial (see eqn (28)) has no linear term, the possible poles of $\hat{\sigma}_{ij}(s, \theta)$ are independent of the material properties. They are $s = -2, -3, -4, \ldots, -n, \ldots$.

For the case II : $\breve{B}_2 \neq 0$, there is

$$\Delta = 4\breve{B}_2\{(2-m)\cos[(s+1)\pi] + 4s^2 + 2sm + 8s + 3m + 2\} - 8(s+2)mG_0 \sin[(s+1)\pi]. \tag{82}$$

The solution of $\Delta = 0$ is dependent on the value of $\breve{B}_2$, m (the Poisson's ratio v and the stress state (plane stress or plane strain)) and G_0 (the shear modulus at the interface).

To show the agreement of the stresses calculated from FEM and eqn (78), the following four examples will be given, which are

$$(A)\left.\frac{\partial G_2}{\partial \theta}\right|_{\theta=0} = 0 \quad \text{and} \quad \left.\frac{\partial \hat{T}_{2n}(s+2,\theta)}{\partial \theta}\right|_{\theta=0} = 0$$

$$(B)\left.\frac{\partial G_2}{\partial \theta}\right|_{\theta=0} = 0 \quad \text{and} \quad \left.\frac{\partial \hat{T}_{2n}(s+2,\theta)}{\partial \theta}\right|_{\theta=0} \neq 0$$

$$(C)\left.\frac{\partial G_2}{\partial \theta}\right|_{\theta=0} \neq 0 \quad \text{and} \quad \left.\frac{\partial \hat{T}_{2n}(s+2,\theta)}{\partial \theta}\right|_{\theta=0} = 0$$

$$(D)\left.\frac{\partial G_2}{\partial \theta}\right|_{\theta=0} \neq 0 \quad \text{and} \quad \left.\frac{\partial \hat{T}_{2n}(s+2,\theta)}{\partial \theta}\right|_{\theta=0} \neq 0.$$

Case (A) is corresponding to that in the profile of the Young's modulus E and in the thermal expansion coefficient α there is no linear term. Case (B) is corresponding to that in the profile of the Young's modulus E there is no linear term, but in the thermal expansion coefficient α there is a linear term. Case (C) is corresponding to that in the profile of the Young's modulus E there is a linear term, but in the thermal expansion coefficient α there is no linear term. Case (D) is corresponding to that in the profile of the Young's modulus E there is a linear term, and also in the thermal expansion coefficient α there is a linear term. From eqn (30) it can be seen that all combinations $(\breve{B} = 0, \breve{B} \neq 0, \partial \hat{T}_n(s+2, \theta)/\partial\theta\,|_{\theta=0} = 0, \partial \hat{T}_n(s+2, \theta)/\partial\theta\,|_{\theta=0} \neq 0)$ are considered in the cases (A)–(D).

EXAMPLES AND DISCUSSIONS

Four examples, in which all special cases are considered, will be given to show the agreement of the stresses calculated by FEM and with eqn (78). For all the examples the thermal loading is a homogeneous temperature change of $T_0 = -100$ K and the geometry is $\theta_1 = -\theta_2 = 90°$, $H_1/L = H_2/L = 2$ (see Fig. 1b). It is assumed that material 1 is a homogeneous material and material 2 is a graded material. The results are for plane strain. The only difference between the examples is that the material properties and the profile for E and α are different.

Equation (78) is deduced for the case of a semi-infinite joint with the temperature change:

$$T = \begin{cases} T_0 & \text{for } r \leqslant R_0 \\ 0 & \text{for } r > R_0 \end{cases}$$

but it can be used for a finite joint with a homogeneous temperature change to calculate the stresses near the free edge of the interface in a joint with a graded material. This means that the angular functions $F_{ijnl}(\theta)$, $g_{ijn}(\theta)$ and $h_{ijn}(\theta)$ in eqns (74)–(76) are the same for a finite joint as for a semi-infinite joint, only for a finite joint the quantity R_0 is unknown, therefore, the factors K_{nl} in eqn (76) are unknown. They have to be determined from the stresses calculated by FEM. The method to determine the factors K_{nl} for a finite joint will be simply presented as follows. In eqns (78) and (76) the quantities $F_{ijnl}(\theta)$, $g_{ijn}(\theta)$, $h_{ijn}(\theta)$ and ω_n can be calculated analytically, if the stresses are known from the FEM, we can define one quantity Π

$$\Pi_{ij} = \sum_{l=1}^{M} \left\{ \sigma_{ij}^{\mathrm{FEM}}(r_l, \theta_l) - \sum_{n=1}^{N} \{(r_l/L)^{\omega_n}[K_{n1}F_{ijn1}(\theta_l) + K_{n2}F_{ijn2}(\theta_l) \right.$$
$$\left. + K_{n3}F_{ijn3}(\theta_l) + \cdots + g_{ijn}(\theta_l)\ln(r_l/L) + h_{ijn}(\theta_l)]\} \right\}^2 \quad (83)$$

where there is $ij = xx$, yy, xy, or rr, $\theta\theta$, $r\theta$, M is the number of the used points for the determination of the K_{nl} factors. In principle, any stress component at any point (r_l, θ_l) near the free edge of the interface can be used. In general, we use the points along a line, i.e. θ_l is a constant. Following the least square method the factors K_{nl} can be determined from

$$\frac{\partial \Pi_{ij}}{\partial K_{nl}} = 0. \quad (84)$$

For the calculation of the stresses from FEM a standard element with eight nodes is used. The mesh needs not very fine. In the FE-code program ABAQUS it is possible to give the material properties in the graded material as a continuous function by using the subroutine UMAT.

Example 1

The materials data for example 1 are

$$E_1 = 100\,\text{GPa} \quad \nu_1 = 0.25 \quad \alpha_1 = 2.5 * 10^{-6}/\text{K}$$

$$E_2 = 100 + 50\bar{y}^2 = 100 + 50\bar{r}^2 \sin^2(\theta)\,\text{GPa}$$

$$\alpha_2 = [2.5 + 5\bar{y}^2] * 10^{-6}/\text{K} = [2.5 + 5\bar{r}^2 \sin^2(\theta)] * 10^{-6}/\text{K}$$

$$\nu_2 = \nu_1.$$

For this example we have $\partial G_2/\partial\theta\,|_{\theta=0} = 0$ (i.e. $\check{B}_2 = 0$) and $\partial \hat{T}_{2n}(s+2,\theta)/\partial\theta\,|_{\theta=0} = 0$. This is the case I, i.e. the poles of $\hat{\sigma}_{ij}(s,\theta)$ are independent of the value of ν_2.

The poles of $\hat{\sigma}_{ij}(s,\theta)$ are $s = -2, -3, -4, -5, -6, \ldots$. To calculate the stresses from eqn (78) five terms are used, where $s = -2, -3, -5$ are the poles of the first-order of $\hat{\sigma}_{ij}(s,\theta)$ and $s = -4, -6$ are the poles of the second-order of $\hat{\sigma}_{ij}(s,\theta)$.

To check eqn (78) the quantities used in eqn (78) to calculate the stresses near the free edge of the interface in a joint with a graded material are given as follows for different components along different directions:

for $\theta = 0°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0	0	0
	2	1	0	0	0
	3	2	0.04053	0	−0.03125
	4	3	−0.0252	0	0
	5	4	0	0	0.048
$\sigma_{\theta\theta}$	1	0	0.0246	0	0
	2	1	−0.0696	0	0
	3	2	−0.081	0	0.1042
	4	3	0.051	0	0
	5	4	0	0	−0.08021
$\sigma_{r\theta}$	1	0	0	0	0
	2	1	−0.0496	0	0
	3	2	−0.0325	0.0531	0.0221
	4	3	−0.0921	0	0
	5	4	0.082	−0.042	−0.0078

for $\theta = 90°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0246	0	0
	2	1	−0.0496	0	0
	3	2	0.0405	0	0.01042
	4	3	−0.007	0	0
	5	4	0	0	0.00104

for $\theta = 45°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0123	0	0
	2	1	−0.0071	0	0
	3	2	−0.0405	0	0.02368
	4	3	0.0508	0	0
	5	4	−0.041	0.021	−0.004
$\sigma_{\theta\theta}$	1	0	0.0123	0	0
	2	1	−0.0772	0	0
	3	2	0.073	−0.05305	−0.01926
	4	3	−0.0359	0	0
	5	4	0.041	−0.021	0.02
$\sigma_{r\theta}$	1	0	0.0123	0	0
	2	1	−0.0421	0	0
	3	2	−0.0021	−0.0256	0.0046
	4	3	0.0653	0	0
	5	4	−0.041	0.021	0.019

for $\theta = -45^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0123	0	0
	2	1	−0.0421	0	0
	3	2	−0.0405	0	0.0597
	4	3	−0.0142	0	0
	5	4	0.041	−0.021	−0.024
$\sigma_{\theta\theta}$	1	0	0.0123	0	0
	2	1	0.0246	0	0
	3	2	0.008	0.05305	−0.0224
	4	3	0.0359	0	0
	5	4	−0.041	0.021	0.081
$\sigma_{r\theta}$	1	0	−0.0123	0	0
	2	1	−0.0175	0	0
	3	2	0.0231	−0.0256	−0.0475
	4	3	0.0653	0	0
	5	4	−0.041	0.021	−0.04

for $\theta = -90^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0246	0	0
	2	1	0.0496	0	0
	3	2	0.0405	0	−0.07292
	4	3	0.007	0	0
	5	4	0	0	−0.032

Comparisons of the stresses obtained by FEM and with eqn (78) are given in Figs 2–6 for different components along different directions. It is shown that when five terms are used in eqn (78) for $r/L \leq 0.1$ they are in good agreement. For the range of $r/L > 0.1$ the stresses calculated with eqn (78) deviate from those of FEM. It is due to the effect of the higher-order terms on the stresses.

Example 2

The materials data for example 2 are

$$E_1 = 100\,\text{GPa} \quad \nu_1 = 0.25 \quad \alpha_1 = 2.5 * 10^{-6}/\text{K}$$
$$E_2 = 100 + 50\bar{y}^2 = 100 + 50\bar{r}^2 \sin^2(\theta)\,\text{GPa}$$
$$\alpha_2 = [2.5 - 5\bar{y}] * 10^{-6}/\text{K} = [2.5 + 5\bar{r}\sin(\theta)] * 10^{-6}/\text{K}$$
$$\nu_2 = \nu_1$$

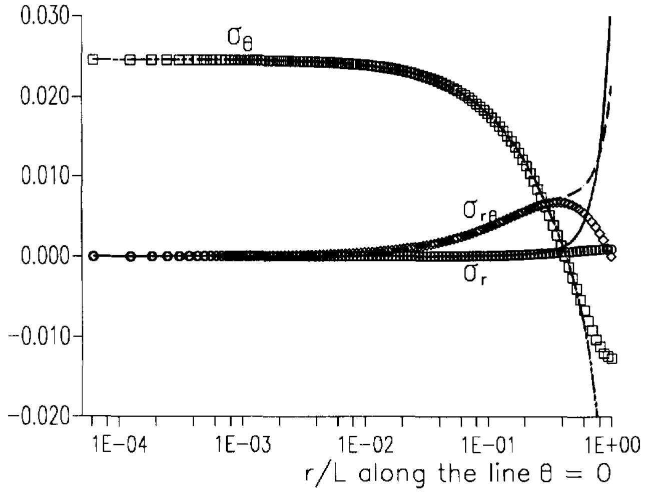

Fig. 2. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 0$ for example 1.

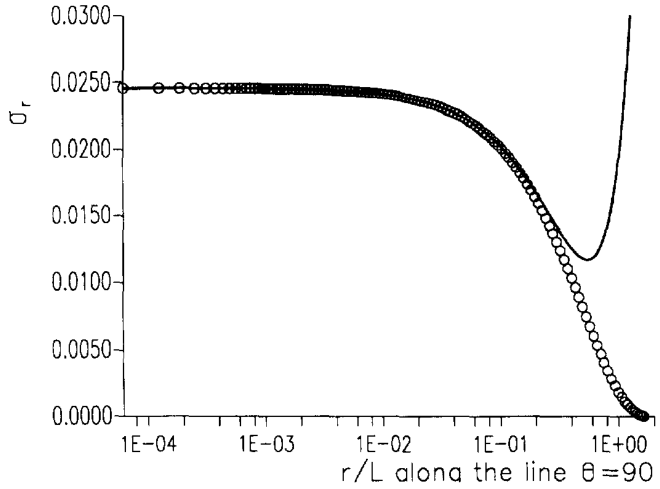

Fig. 3. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 90$ for example 1.

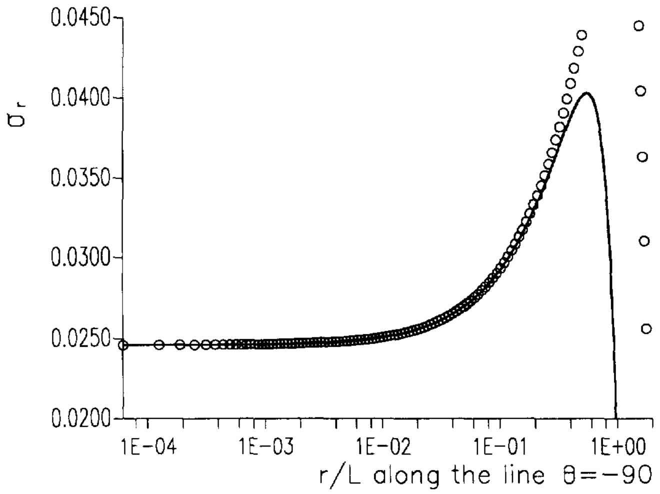

Fig. 4. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -90$ for example 1.

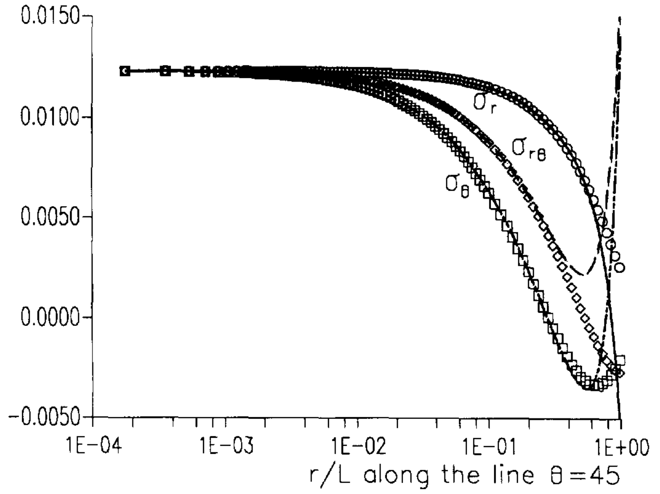

Fig. 5. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 45$ for example 1.

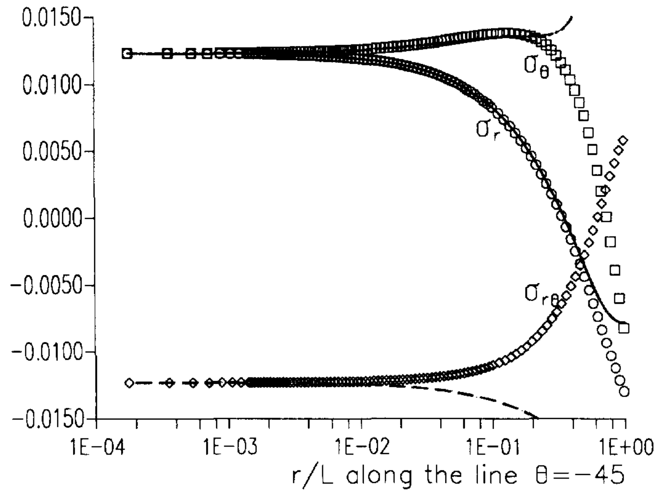

Fig. 6. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -45$ for example 1.

where the angle θ is negative. For this example we have $\partial G_2/\partial\theta\,|_{\theta=0} = 0$ (i.e. $\check{B}_2 = 0$) and $\partial \hat{T}_{2n}(s+2,\theta)/\partial\theta\,|_{\theta=0} \neq 0$. This is the case I, i.e. the poles of $\hat{\sigma}_{ij}(s,\theta)$ are independent of the value of v_2, but the right-hand side of eqn (30) is different as that for example 1. The poles of $\hat{\sigma}_{ij}(s,\theta)$ are $s = -2, -3, -4, -5, \ldots$. To calculate the stresses from eqn (78) four terms are used, where $s = -2$ is the pole of the first-order of $\hat{\sigma}_{ij}(s,\theta)$ and $s = -3, -4, -5$ are the poles of the second-order of $\hat{\sigma}_{ij}(s,\theta)$.

The quantities used in eqn (78) to calculate the stresses near the free edge of the interface in a joint with a graded material are as follows for different components along different directions:

for $\theta = 0°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0	0	0
	2	1	0	0	−0.0212
	3	2	0.025	0	−0.006255
	4	3	−0.037	0.0212	0.008848
$\sigma_{\theta\theta}$	1	0	0.0226	0	0
	2	1	−0.0297	0.0424	0.0212
	3	2	−0.0499	0	0.0208
	4	3	0.0635	−0.0424	0.0071
$\sigma_{r\theta}$	1	0	0	0	0
	2	1	0.0178	0	−0.0167
	3	2	0.0246	0.0106	0.0044
	4	3	−0.0521	0	0.0458

for $\theta = 90°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0226	0	0
	2	1	−0.0178	0	−0.01667
	3	2	0.025	0	0.0021
	4	3	−0.0261	0	−0.00208

for $\theta = -90^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0226	0	0
	2	1	0.0178	0	−0.05
	3	2	0.025	0	−0.0146
	4	3	0.0261	0	−0.0312

for $\theta = 45^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0113	0	0
	2	1	−0.0041	0.015	0.0059
	3	2	−0.025	0	0.0047
	4	3	0.0409	−0.015	0.0107
$\sigma_{\theta\theta}$	1	0	0.0113	0	0
	2	1	−0.0288	0.015	0.0059
	3	2	−0.0015	−0.0106	−0.0039
	4	3	−0.0449	0.03	−0.00089
$\sigma_{r\theta}$	1	0	0.0113	0	0
	2	1	−0.0165	0.015	0.0091
	3	2	−0.0257	−0.0053	0.000915
	4	3	0.0369	0	−0.0147

for $\theta = -45^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0113	0	0
	2	1	−0.165	0.015	0.0177
	3	2	−0.025	0	0.0119
	4	3	0.0409	−0.015	0.0158
$\sigma_{\theta\theta}$	1	0	0.0113	0	0
	2	1	0.0082	0.015	−0.0295
	3	2	0.0514	0.0106	−0.0045
	4	3	−0.0449	0.03	0.0286
$\sigma_{r\theta}$	1	0	−0.0113	0	0
	2	1	0.0041	−0.015	0.0027
	3	2	−0.000728	−0.0053	−0.0095
	4	3	0.0369	0	−0.0442

Comparisons of the stresses obtained by FEM and with eqn (78) are given in Figs 7–11 for different components along different directions. It is shown that when four terms are used in eqn (78) for $r/L \leqslant 0.1$ they also are in good agreement.

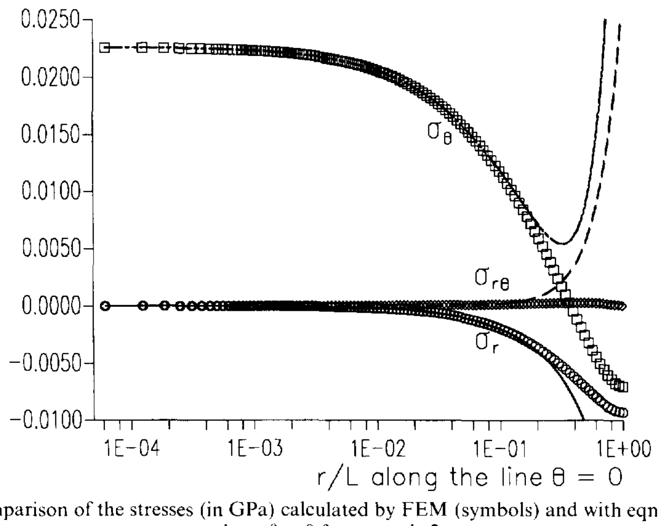

Fig. 7. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 0$ for example 2.

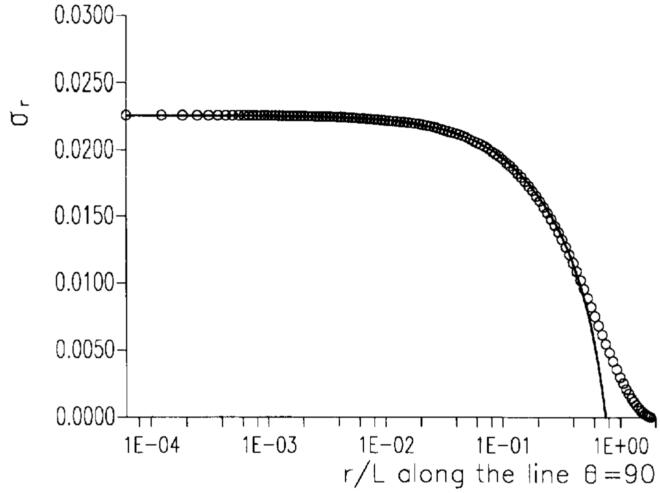

Fig. 8. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 90$ for example 2.

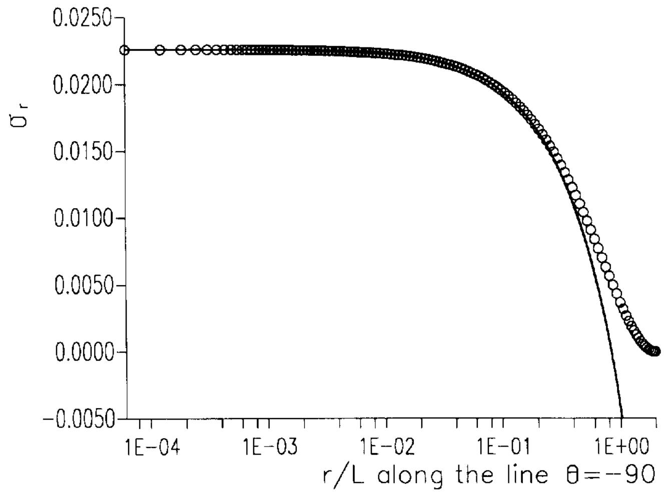

Fig. 9. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -90$ for example 2.

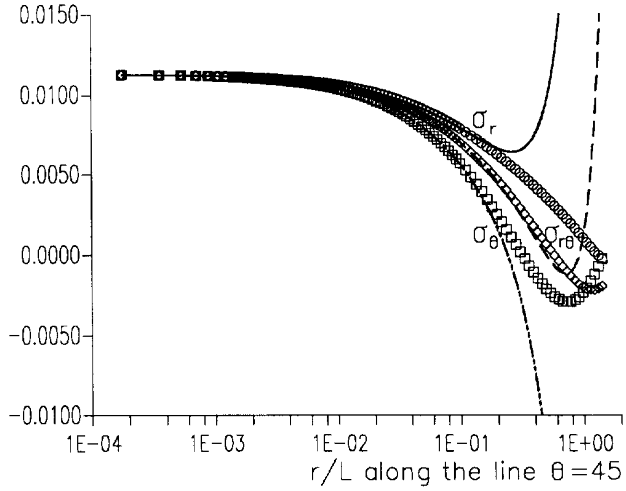

Fig. 10. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 45$ for example 2.

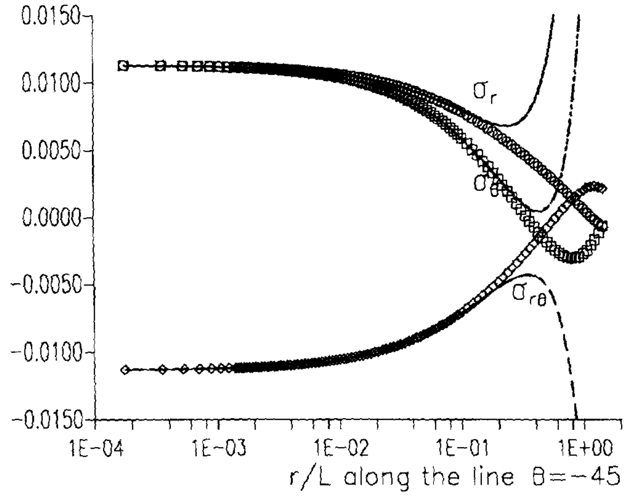

Fig. 11. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -45$ for example 2.

Example 3

The materials data for example 3 are

$$E_1 = 100\,\mathrm{GPa} \quad \nu_1 = \tfrac{1}{3} \quad \alpha_1 = 2.5*10^{-6}/\mathrm{K}$$

$$E_2 = 100-50\bar{y} = 100+50\bar{r}\sin(\theta)\,\mathrm{GPa}$$

$$\alpha_2 = [2.5+5\bar{y}^2]*10^{-6}/\mathrm{K} = [2.5+5\bar{r}^2\sin^2(\theta)]*10^{-6}/\mathrm{K}$$

$$\nu_2 = \nu_1$$

where the angle θ is negative.

For this example we have $\partial G_2/\partial\theta\,|_{\theta=0} \neq 0$ (i.e. $\breve{B}_2 \neq 0$) and $\partial\hat{T}_{2n}(s+2,\theta)/\partial\theta\,|_{\theta=0} = 0$. This is the case II, i.e. the poles of $\hat{\sigma}_{ij}(s,\theta)$ depend on the value of ν_2. The poles of $\hat{\sigma}_{ij}(s,\theta)$ are $s = -2, -3, -4, -4.1529, -5, \ldots,$ where $s = -2$, $s = -4.1529$ and $s = -5$ are the poles of the first-order of $\hat{\sigma}_{ij}(s,\theta)$, and $s = -3$ and $s = -4$ are the poles of the second-order of $\hat{\sigma}_{ij}(s,\theta)$. To calculate the stresses from eqn (78) only three terms are used. Because the difference of $s = -4$ and $s = -4.1529$ is small, it is difficult to determine, accurately, the corresponding factors K_{nl} for $s = -4$ and $s = -4.1529$ at the same time. Therefore, either $s = -2, -3, -4$ or $s = -2, -3, -4.1529$ are used to calculate the stresses from eqn (78).

If the poles $s = -2, -3, -4.1529$ are used, the quantities applied in eqn (78) to calculate the stresses near the free edge of the interface in a joint with a graded material are as follows for different components along different directions:

for $\theta = -90^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0284	0	0
	2	1	0.06364	0	−0.006806
	3	2.1529	0.0711	0	0

for $\theta = 0^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0	0	0
	2	1	0	0	−0.02563
	3	2.1529	−14.57	0	0
$\sigma_{\theta\theta}$	1	0	0.0284	0	0
	2	1	−0.0509	0.002698	0.001628
	3	2.1529	0.07522	0	0
$\sigma_{r\theta}$	1	0	0	0	0
	2	1	0.06364	0	−0.004688
	3	2.1529	0	0	0

for $\theta = -45^\circ$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.0142	0	0
	2	1	−0.04166	0.0009539	0.002505
	3	2.1529	−0.1698	0	0
$\sigma_{\theta\theta}$	1	0	0.0142	0	0
	2	1	0.04791	0.0009539	−0.005622
	3	2.1529	0.2792	0	0
$\sigma_{r\theta}$	1	0	−0.0142	0	0
	2	1	−0.003	−0.0009539	0.001354
	3	2.1529	−0.03074	0	0

Comparisons of the stresses obtained by FEM and with eqn (78) are given in Figs 12–14 for different components along different directions. It is shown that when three terms with $s = -2, -3, -4.1529$ are used in eqn (78) for $r/L \leqslant 0.01$ they are in good agreement.

If the poles $s = -2, -3, -4$ are used, comparisons of the stresses obtained by FEM and with eqn (78) are given in Figs 15–17 for different components along different directions. It is shown that when $s = -2, -3, -4$ are used in eqn (78) for $r/L \leqslant 0.01$ they also exhibit a good agreement. By comparing Figs 12 and 15, Figs 13 and 16, Figs 14 and 17, we can see that the agreement of the stresses obtained by FEM and with eqn (78) from three terms with $s = -2, -3, -4$ is much better than that with $s = -2, -3, -4.1529$.

Example 4

The materials data for example 4 are

$$E_1 = 100\,\text{GPa} \quad \nu_1 = \tfrac{1}{3} \quad \alpha_1 = 2.5 * 10^{-6}/\text{K}$$
$$E_2 = 100 - 50\bar{y} = 100 + 50\bar{r}\sin(\theta)\,\text{GPa}$$
$$\alpha_2 = [2.5 - 5\bar{y}] * 10^{-6}/\text{K} = [2.5 + 5\bar{r}\sin(\theta)] * 10^{-6}/\text{K}$$
$$\nu_2 = \nu_1$$

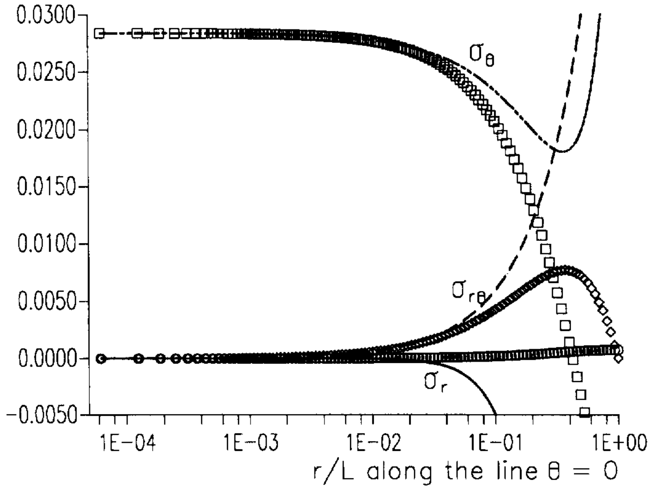

Fig. 12. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 0$ for example 3 with $\omega_n = 0, 1, 2.1529$.

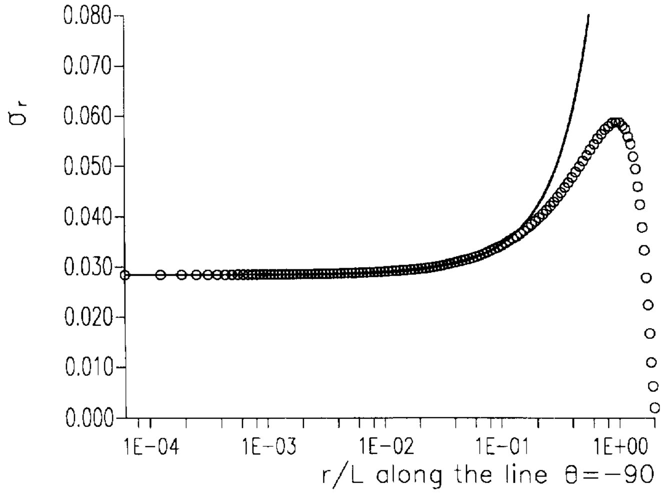

Fig. 13. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -90$ for example 3 with $\omega_n = 0, 1, 2.1529$.

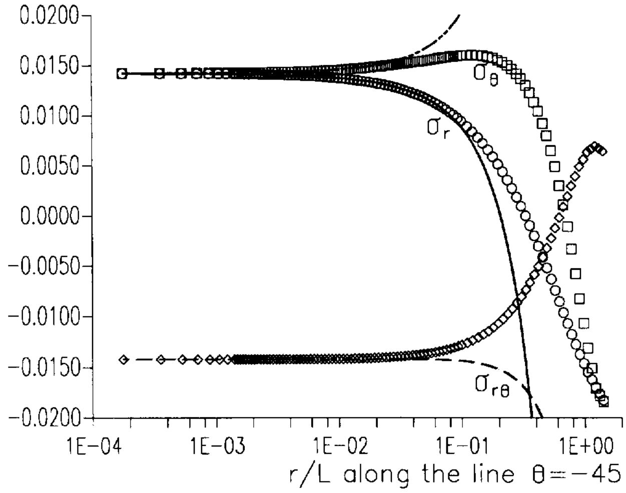

Fig. 14. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -45$ for example 3 with $\omega_n = 0, 1, 2.1529$.

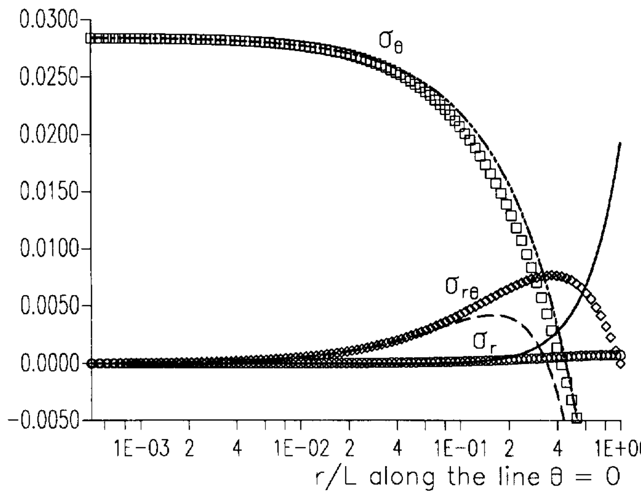

Fig. 15. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 0$ for example 3 with $\omega_n = 0, 1, 2$.

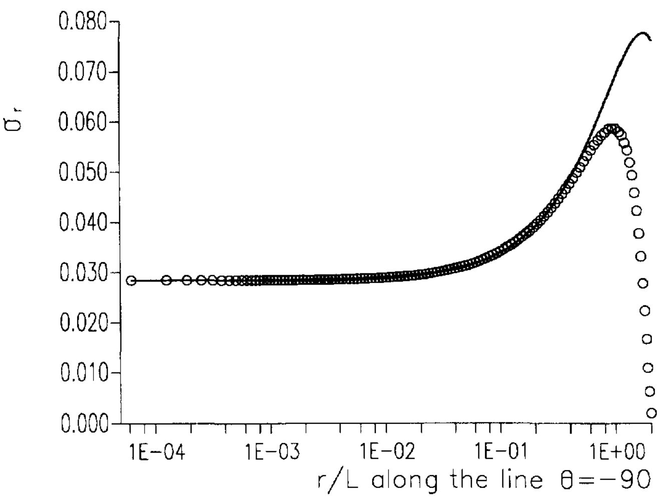

Fig. 16. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -90$ for example 3 with $\omega_n = 0, 1, 2$.

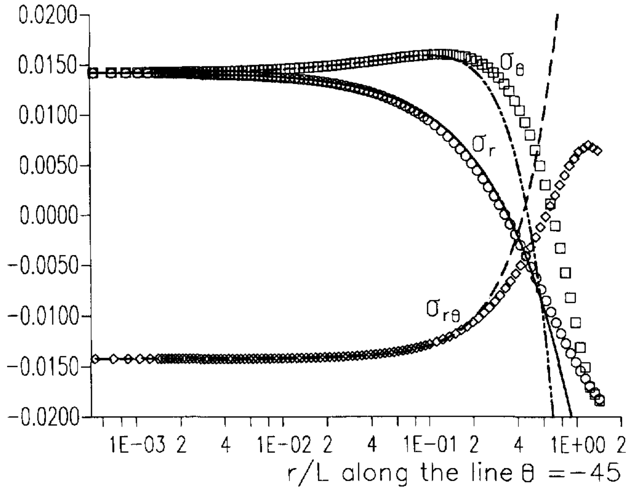

Fig. 17. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -45$ for example 3 with $\omega_n = 0, 1, 2$.

where the angle θ is negative. For this example we have $\partial G_2/\partial\theta\,|_{\theta=0} \neq 0$ (i.e. $\breve{B}_2 \neq 0$) and $\partial \hat{T}_{2n}(s+2, \theta)/\partial\theta\,|_{\theta=0} \neq 0$. This is the case II, i.e. the poles of $\hat{\sigma}_{ij}(s, \theta)$ depend on the value of v_2, which are $s = -2, -3, -4, -4.1529, -5, \ldots$. However, the right-hand side of eqn (30) is different as that for example 3. As in example 3, only three terms are used to calculate the stresses from eqn (78), where $s = -2, -3, -4.1529$. $s = -2$ and $s = -4.1529$ are the poles of the first-order of $\hat{\sigma}_{ij}(s, \theta)$ and $s = -3$ is the pole of the second-order of $\hat{\sigma}_{ij}(s, \theta)$. The quantities used in eqn (78) to calculate the stresses near the free edge of the interface of a joint with a graded material are as follows for different components along different directions:

for $\theta = 90°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.02594	0	0
	2	1	−0.0262	0	−0.01682
	3	2.1529	−0.02203	0	0

for $\theta = 0°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0	0	0
	2	1	0	0	−0.02563
	3	2.1529	1.719	0	0
$\sigma_{\theta\theta}$	1	0	0.02597	0	0
	2	1	−0.008242	0.05126	0.02083
	3	2.1529	−0.5154	0	0
$\sigma_{r\theta}$	1	0	0	0	0
	2	1	0.0262	0	−0.02344
	3	2.1529	0	0	0

for $\theta = 45°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.01297	0	0
	2	1	−0.006335	0.01812	0.004249
	3	2.1529	−0.01149	0	0
$\sigma_{\theta\theta}$	1	0	0.01297	0	0
	2	1	−0.0307	0.01812	0.008927
	3	2.1529	0.01562	0	0
$\sigma_{r\theta}$	1	0	0.01297	0	0
	2	1	−0.0122	0.01812	0.01048
	3	2.1529	−0.0067	0	0

for $\theta = -45°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.01297	0	0
	2	1	−0.0122	0.01812	0.02082
	3	2.1529	−0.01149	0	0
$\sigma_{\theta\theta}$	1	0	0.01297	0	0
	2	1	0.0249	0.01812	−0.04079
	3	2.1529	0.01562	0	0
$\sigma_{r\theta}$	1	0	−0.01297	0	0
	2	1	−0.006335	−0.01812	0.006069
	3	2.1529	0.0067	0	0

for $\theta = -90°$

σ_{ij}	n	ω_n	$f_{ijn}(\theta, K_n)$, GPa	$g_{ijn}(\theta)$, GPa	$h_{ijn}(\theta)$, GPa
σ_{rr}	1	0	0.02594	0	0
	2	1	0.0262	0	−0.0637
	3	2.1529	−0.02203	0	0

Comparisons of the stresses obtained by FEM and with eqn (78) are given in Figs 18–22 for different components along different directions. It is shown that when three terms with $s = -2, -3, -4.1529$ are used in eqn (78) for $r/L \leqslant 0.01$ they are in good agreement. If three terms with $s = -2, -3, -4$ are used in eqn (78), the agreement is much better. From examples 1 to 4 it can be seen that if three terms in eqn (78) are considered the poles $s = -2, -3, -4$ can be always used to describe the stresses in the range of $r/L \leqslant 0.01$ very well.

CONCLUSIONS

In this paper a method has been presented to describe analytically the stress distribution near the free edge of the interface in a joint with a graded material under a thermal loading. The stresses near the free edge of the interface in a joint with a graded material can be described by

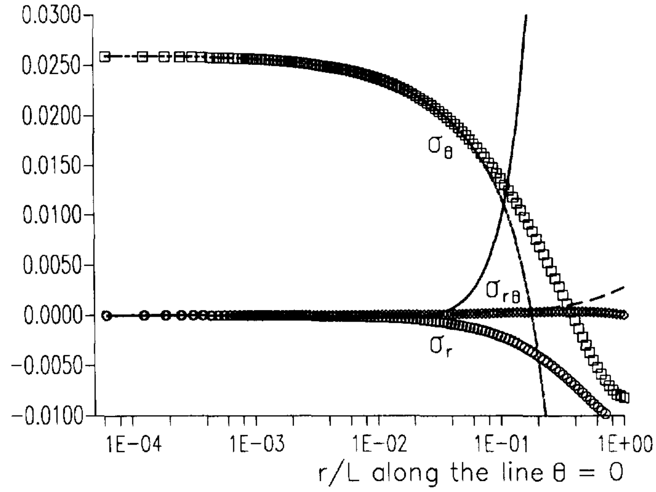

Fig. 18. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 0$ for example 4.

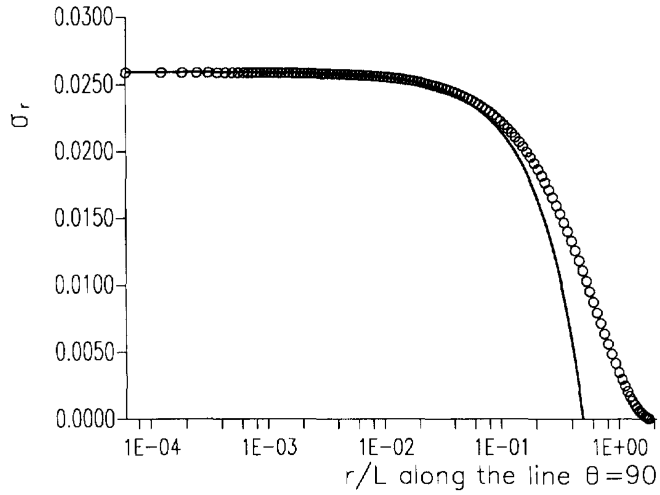

Fig. 19. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 90$ for example 4.

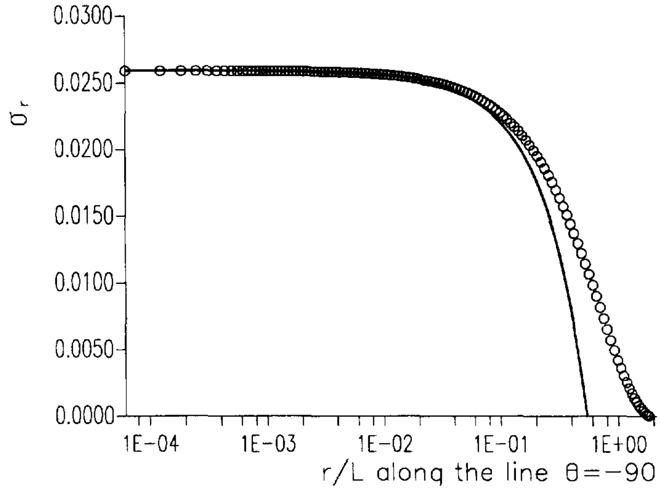

Fig. 20. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -90$ for example 4.

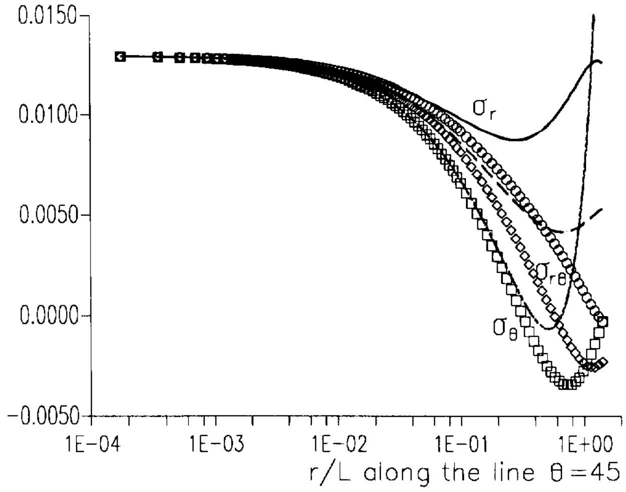

Fig. 21. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = 45$ for example 4.

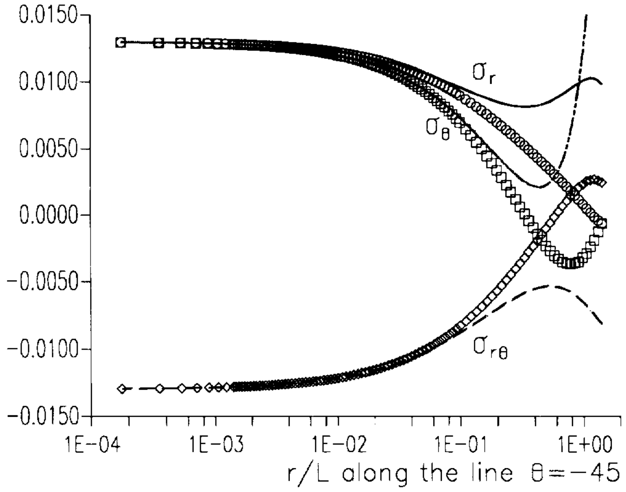

Fig. 22. Comparison of the stresses (in GPa) calculated by FEM (symbols) and with eqn (78) (lines) along $\theta = -45$ for example 4.

$$\sigma_{ij}(r,\theta) = \sum_{n=1}^{N} \{(r/L)^{\omega_n}[f_{ijn}(\theta, K_n) + g_{ijn}(\theta)\ln(r/L) + h_{ijn}(\theta)]\} \tag{85}$$

with

$$f_{ijn}(\theta, K_n) = K_{n1}F_{ijn1}(\theta) + K_{n2}F_{ijn2}(\theta) + K_{n3}F_{ijn3}(\theta) + \cdots \tag{86}$$

where the angular functions $F_{ijnl}(\theta)$ $(l = 1, 2, 3, \ldots)$, $g_{ijn}(\theta)$ and $h_{ijn}(\theta)$ and the stress exponent ω_n can be determined analytically. They are a function of the profile of the shear modulus G and the thermal expansion coefficient α, and also a function of the value of the Poisson's ratio v and the geometry (angles θ_1, θ_2). For a finite joint the factors K_{nl} are unknown. They have to be determined from the stresses calculated by FEM. The functions $F_{ijnl}(\theta)$, $g_{ijn}(\theta)$ and $h_{ijn}(\theta)$ have the unit of the stress, i.e. MPa or GPa, and K_{nl} is dimensionless. The stresses $\sigma_{ij}(r, \theta)$ are proportional to the temperature change T_0.

In a quarter-planes joint for the case, in which the graded material has the profile of a polynomial and it is only dependent on the coordinate y, as long as the polynomial in the Young's modulus E has no linear term, the stress exponent ω_n is independent of the material property v and G_0, and there is always $\omega_n = 0, 1, 2, 3, \ldots, n, \ldots$. If the polynomial in the

Young's modulus E has a linear term, the stress exponent ω_n is dependent on the material property v, G_0 and the stress state (plane stress or plane strain), the stress exponents are $\omega_n = 0, 1, 2, 3, \ldots, n, \ldots$ plus those which are dependent on the material properties v, G_0 and the stress state.

Four examples for different cases have been presented for a finite joint with a graded material. Comparisons of the stresses obtained by FEM and from the analytical description have shown that if three terms in eqn (78) are used for $r/L \leqslant 0.01$ they are in good agreement. In the case of a polynomial without a linear term, good agreement is obversed even for $r/L \leqslant 0.1$.

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